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ON THE PROBLEM OF THREE BODIES IN THE PLANE

BY

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Synopsis.

Three bodies with finite masses are assumed to move in a plane, subject to Newton's law of gravitation. By the introduction of suitable auxiliary variables the equations of motion are transformed into a system of differential equations of the second degree, permitting to expand the unknown quantities in powers of the time t, the coefficients of t^{ν} being calculated by means of a set of recurrence formulas. Sufficient conditions for the convergence of the resulting series are given, and the practical working of the method is illustrated by a numerical example.

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1. On a former occasion[1] I have shown how a particular case of the Problem of Three Bodies can be dealt with by transforming the equations of motion into a system of differential equations of the second degree in the unknown variables, permitting to expand these in powers of the time t, the coefficients of t^ν being calculated by a set of recurrence formulas. The same method can, in principle, be employed in other cases of the dynamical astronomy, and I propose in the present paper to extend it to the problem of three finite bodies moving in the same fixed plane and subject to Newton's law of gravitation. The number of recurrence formulas naturally increases, but without becoming unwieldy, as will be shown by a numerical example.

Let the three masses be m_1, m_2 and m_3, the coordinates of m_i being (x_i, y_i), and let us put

$$\left.\begin{aligned} r_1^2 &= (x_2 - x_3)^2 + (y_2 - y_3)^2 \\ r_2^2 &= (x_3 - x_1)^2 + (y_3 - y_1)^2 \\ r_3^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 \end{aligned}\right\} \quad (1)$$

so that r_1 is the distance between m_2 and m_3, etc.

Then the equations of motion are[2]

$$\left.\begin{aligned} \frac{d^2 x_1}{dt^2} &= m_2 \frac{x_2 - x_1}{r_3^3} + m_3 \frac{x_3 - x_1}{r_2^3} \\ \frac{d^2 x_2}{dt^2} &= m_3 \frac{x_3 - x_2}{r_1^3} + m_1 \frac{x_1 - x_2}{r_3^3} \\ \frac{d^2 x_3}{dt^2} &= m_1 \frac{x_1 - x_3}{r_2^3} + m_2 \frac{x_2 - x_3}{r_1^3} \end{aligned}\right\} \quad (2)$$

and corresponding equations with y instead of x.

[1] J. F. Steffensen: 'On the Restricted Problem of Three Bodies'. Mat. Fys. Medd. Dan. Vid. Selsk. **30**, no. 18 (1956).

[2] It is assumed throughout that none of the distances r_1, r_2, r_3 vanishes.

We now introduce for $i = 1, 2, 3$ the auxiliary variables

$$\varrho_i = r_i^2, \qquad \sigma_i = r_i^{-3}, \tag{3}$$

so that

$$2\varrho_i \frac{d\sigma_i}{dt} + 3\sigma_i \frac{d\varrho_i}{dt} = 0, \tag{4}$$

$$\left.\begin{aligned} \varrho_1 &= (x_2 - x_3)^2 + (y_2 - y_3)^2 \\ \varrho_2 &= (x_3 - x_1)^2 + (y_3 - y_1)^2 \\ \varrho_3 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 \end{aligned}\right\} \tag{5}$$

while the equations of motion become

$$\left.\begin{aligned} \frac{d^2 x_1}{dt^2} &= m_2\ (x_2 - x_1)\ \sigma_3 + m_3\ (x_3 - x_1)\ \sigma_2 \\ \frac{d^2 x_2}{dt^2} &= m_3\ (x_3 - x_2)\ \sigma_1 + m_1\ (x_1 - x_2)\ \sigma_3 \\ \frac{d^2 x_3}{dt^2} &= m_1\ (x_1 - x_3)\ \sigma_2 + m_2\ (x_2 - x_3)\ \sigma_1 \end{aligned}\right\} \tag{6}$$

and corresponding equations with y instead of x.

For the determination of the 12 unknowns x_i, y_i, σ_i, ϱ_i we now have the 12 equations (4), (5), (6) and the corresponding equations in y, which are all of the second degree in the unknowns and can be treated in the way indicated above.

2. If, however, only the distances of the masses from each other at any given time are required, the number of equations can be reduced to 10. In that case the absolute positions in the plane can be determined afterwards, if desired. This is the relativistic point of view, familiar from the treatment of the Restricted Problem of Three Bodies. In the present case we put

$$\left.\begin{aligned} \xi_1 &= x_2 - x_3, \qquad & \xi_2 &= x_3 - x_1 \\ \eta_1 &= y_2 - y_3, \qquad & \eta_2 &= y_3 - y_1 \end{aligned}\right\} \tag{7}$$

and for abbreviation

$$M_1 = m_2 + m_3, \qquad M_2 = m_1 + m_3. \tag{8}$$

We then obtain from (5)

$$\left.\begin{aligned} \varrho_1 &= \xi_1^2 + \eta_1^2, \qquad \varrho_2 = \xi_2^2 + \eta_2^2 \\ \varrho_3 &= \varrho_1 + \varrho_2 + 2\ \xi_1\ \xi_2 + 2\eta_1\ \eta_2 \end{aligned}\right\} \tag{9}$$

and from (6)

$$\left.\begin{aligned}\frac{d^2\xi_1}{dt^2} &= m_1\,(\xi_2\,\sigma_2 - \xi_1\,\sigma_3 - \xi_2\,\sigma_3) - M_1\,\xi_1\,\sigma_1\\ \frac{d^2\xi_2}{dt^2} &= m_2\,(\xi_1\,\sigma_1 - \xi_1\,\sigma_3 - \xi_2\,\sigma_3) - M_2\,\xi_2\,\sigma_2\end{aligned}\right\}\quad(10)$$

and, replacing ξ by η in this,

$$\left.\begin{aligned}\frac{d^2\eta_1}{dt^2} &= m_1\,(\eta_2\,\sigma_2 - \eta_1\,\sigma_3 - \eta_2\,\sigma_3) - M_1\,\eta_1\,\sigma_1\\ \frac{d^2\eta_2}{dt^2} &= m_2\,(\eta_1\,\sigma_1 - \eta_1\,\sigma_3 - \eta_2\,\sigma_3) - M_2\,\eta_2\,\sigma_2\,.\end{aligned}\right\}\quad(11)$$

(4) and (9)–(11) are 10 equations for determining the 10 unknowns $\xi_1,\ \xi_2,\ \eta_1,\ \eta_2,\ \varrho_i,\ \sigma_i$. We propose to satisfy them by power series in t, putting

$$\left.\begin{aligned}\xi_1 &= \Sigma\alpha_\nu\,t^\nu, & \xi_2 &= \Sigma\beta_\nu\,t^\nu\\ \eta_1 &= \Sigma\gamma_\nu\,t^\nu, & \eta_2 &= \Sigma\delta_\nu\,t^\nu\end{aligned}\right\}\quad(12)$$

$$\left.\begin{aligned}\varrho_1 &= \Sigma a_\nu\,t^\nu, & \varrho_2 &= \Sigma b_\nu\,t^\nu, & \varrho_3 &= \Sigma c_\nu\,t^\nu\\ \sigma_1 &= \Sigma d_\nu\,t^\nu, & \sigma_2 &= \Sigma e_\nu\,t^\nu, & \sigma_3 &= \Sigma f_\nu\,t^\nu\end{aligned}\right\}\quad(13)$$

the summation being everywhere from $\nu = 0$ to $\nu = \infty$.

Inserting these expansions in the aforesaid equations and demanding that the coefficients of t^n shall vanish, we obtain recurrence formulas for the determination of the coefficients.

We write for abbreviation

$$\varepsilon_\nu = \alpha_\nu + \beta_\nu, \qquad \zeta_\nu = \gamma_\nu + \delta_\nu \quad(14)$$

and for the product-sums

$$(\alpha d)_n = \sum_{\nu=0}^{n} \alpha_\nu\,d_{n-\nu},\ \text{etc.} \quad(15)$$

In this notation we obtain from (10) and (11)

$$\left.\begin{aligned}(n+2)^{(2)}\,\alpha_{n+2} &= m_1\,[(\beta e)_n - (\varepsilon f)_n] - M_1\,(\alpha d)_n\\ (n+2)^{(2)}\,\beta_{n+2} &= m_2\,[(\alpha d)_n - (\varepsilon f)_n] - M_2\,(\beta e)_n\\ (n+2)^{(2)}\,\gamma_{n+2} &= m_1\,[(\delta e)_n - (\zeta f)_n] - M_1\,(\gamma d)_n\\ (n+2)^{(2)}\,\delta_{n+2} &= m_2\,[(\gamma d)_n - (\zeta f)_n] - M_2\,(\delta e)_n\end{aligned}\right\}\quad(16)$$

where $(n+2)^{(2)}$ as usual is short for $(n+2)\,(n+1)$.

Further, we obtain from (9)

$$\left.\begin{aligned} a_n &= (\alpha\alpha)_n + (\gamma\gamma)_n \\ b_n &= (\beta\beta)_n + (\delta\delta)_n \\ c_n &= a_n + b_n + 2\,(\alpha\beta)_n + 2\,(\gamma\delta)_n \end{aligned}\right\} \quad (17)$$

and finally from (4)

$$\left.\begin{aligned} -2\,n\,a_0\,d_n &= \sum_{\nu=0}^{n-1} (3\,n-\nu)\,d_\nu\,a_{n-\nu} \\ -2\,n\,b_0\,e_n &= \sum_{\nu=0}^{n-1} (3\,n-\nu)\,e_\nu\,b_{n-\nu} \\ -2\,n\,c_0\,f_n &= \sum_{\nu=0}^{n-1} (3\,n-\nu)\,f_\nu\,c_{n-\nu}. \end{aligned}\right\} \quad (18)$$

3. The number of constants of integration in (10) and (11), where the σ_i are known functions of the ξ_i and η_i, is only 8 instead of 12 in the original statement of the problem. It is natural to choose as initial values the values of ξ_i, η_i, $\frac{d\xi_i}{dt}$ and $\frac{d\eta_i}{dt}$ for $t = 0$, that is

$$\alpha_0,\, \alpha_1,\, \beta_0,\, \beta_1,\, \gamma_0,\, \gamma_1,\, \delta_0,\, \delta_1. \quad (19)$$

We then obtain first from (17)

$$\left.\begin{aligned} a_0 &= \alpha_0^2 + \gamma_0^2 \\ b_0 &= \beta_0^2 + \delta_0^2 \\ c_0 &= a_0 + b_0 + 2\,\alpha_0\beta_0 + 2\,\gamma_0\delta_0 \end{aligned}\right\} \quad (20)$$

while the relation $\sigma_i^2\,\varrho_i^3 = 1$, resulting from (3), yields, σ_i and ϱ_i being positive,

$$d_0 = \frac{1}{a_0\sqrt{a_0}},\; e_0 = \frac{1}{b_0\sqrt{b_0}},\; f_0 = \frac{1}{c_0\sqrt{c_0}}. \quad (21)$$

After this we find by (17) and (18)

$$\left.\begin{aligned} a_1 &= 2\,(\alpha_0\,\alpha_1 + \gamma_0\,\gamma_1) \\ b_1 &= 2\,(\beta_0\,\beta_1 + \delta_0\,\delta_1) \\ c_1 &= a_1 + b_1 + 2\,(\alpha\beta)_1 + 2\,(\gamma\delta)_1. \end{aligned}\right\} \quad (22)$$

$$-2\,a_0\,d_1 = 3\,d_0\,a_1, \quad -2\,b_0\,e_1 = 3\,e_0\,b_1, \quad -2\,c_0\,f_1 = 3\,f_0\,c_1. \quad (23)$$

The following coefficients are calculated in succession by (16) – (18). The first few of them are

$$\left.\begin{aligned} 2\alpha_2 &= m_1(\beta_0 e_0 - \varepsilon_0 f_0) - M_1 \alpha_0 d_0 \\ 2\beta_2 &= m_2(\alpha_0 d_0 - \varepsilon_0 f_0) - M_2 \beta_0 e_0 \\ 2\gamma_2 &= m_1(\delta_0 e_0 - \zeta_0 f_0) - M_1 \gamma_0 d_0 \\ 2\delta_2 &= m_2(\gamma_0 d_0 - \zeta_0 f_0) - M_2 \delta_0 e_0 . \end{aligned}\right\} \quad (24)$$

$$\left.\begin{aligned} a_2 &= \alpha_1^2 + \gamma_1^2 + 2(\alpha_0\alpha_2 + \gamma_0\gamma_2) \\ b_2 &= \beta_1^2 + \delta_1^2 + 2(\beta_0\beta_2 + \delta_0\delta_2) \\ c_2 &= a_2 + b_2 + 2(\alpha\beta)_2 + 2(\gamma\delta)_2 . \end{aligned}\right\} \quad (25)$$

$$\left.\begin{aligned} -4\, a_0 d_2 &= 6\, d_0 a_2 + 5\, d_1 a_1 \\ -4\, b_0 e_2 &= 6\, e_0 b_2 + 5\, e_1 b_1 \\ -4\, c_0 f_2 &= 6\, f_0 c_2 + 5\, f_1 c_1 . \end{aligned}\right\} \quad (26)$$

$$\left.\begin{aligned} 6\alpha_3 &= m_1[(\beta e)_1 - (\varepsilon f)_1] - M_1(\alpha d)_1 \\ 6\beta_3 &= m_2[(\alpha d)_1 - (\varepsilon f)_1] - M_2(\beta e)_1 \\ 6\gamma_3 &= m_1[(\delta e)_1 - (\zeta f)_1] - M_1(\gamma d)_1 \\ 6\delta_3 &= m_2[(\gamma d)_1 - (\zeta f)_1] - M_2(\delta e)_1 . \end{aligned}\right\} \quad (27)$$

$$\left.\begin{aligned} a_3 &= 2(\alpha_0\alpha_3 + \alpha_1\alpha_2 + \gamma_0\gamma_3 + \gamma_1\gamma_2) \\ b_3 &= 2(\beta_0\beta_3 + \beta_1\beta_2 + \delta_0\delta_3 + \delta_1\delta_2) \\ c_3 &= a_3 + b_3 + 2(\alpha\beta)_3 + 2(\gamma\delta)_3 . \end{aligned}\right\} \quad (28)$$

$$\left.\begin{aligned} -6\, a_0 d_3 &= 9\, d_0 a_3 + 8\, d_1 a_2 + 7\, d_2 a_1 \\ -6\, b_0 e_3 &= 9\, e_0 b_3 + 8\, e_1 b_2 + 7\, e_2 b_1 \\ -6\, c_0 f_3 &= 9\, f_0 c_3 + 8\, f_1 c_2 + 7\, f_2 c_1 . \end{aligned}\right\} \quad (29)$$

4. For the purpose of examining the convergence we put

$$H_\nu = \frac{\lambda^\nu}{(\nu+2)^{(2)}} \quad (\lambda > 0) \quad (30)$$

and

$$s_n = \sum_{\nu=1}^{n} \frac{1}{\nu}. \quad (31)$$

We have then identically

$$\lambda^{-n} H_\nu H_{n-\nu} = \left(\frac{1}{\nu+1} + \frac{1}{n-\nu+1}\right)\frac{1}{(n+3)^{(2)}} - \left(\frac{1}{\nu+2} + \frac{1}{n-\nu+2}\right)\frac{1}{(n+4)^{(2)}} \tag{32}$$

whence

$$\sum_{\nu=0}^{n} H_\nu H_{n-\nu} = 2\lambda^n \frac{2s_{n+1}+n+1}{(n+4)^{(3)}}. \tag{33}$$

$$\sum_{\nu=0}^{n-1} H_\nu H_{n-\nu} = \lambda^n \frac{4s_{n+1}+\frac{3}{2}n-1-\frac{3}{n+1}}{(n+4)^{(3)}} \qquad (n \geqslant 1). \tag{34}$$

$$\sum_{\nu=2}^{n-2} H_\nu H_{n-\nu} = \frac{2}{3}\lambda^n \frac{6s_n+n-10-\frac{12}{n}}{(n+4)^{(3)}} \qquad (n \geqslant 4). \tag{35}$$

Furthermore we have the identity

$$\lambda^{-n} \nu H_\nu H_{n-\nu} = \left(\frac{n+1}{n-\nu+1} - \frac{1}{\nu+1}\right)\frac{1}{(n+3)^{(2)}} - \left(\frac{n+2}{n-\nu+2} - \frac{2}{\nu+2}\right)\frac{1}{(n+4)^{(2)}} \tag{36}$$

whence

$$\sum_{\nu=1}^{n-1} \nu H_\nu H_{n-\nu} = \frac{n\lambda^n}{(n+4)^{(3)}}\left(2s_{n+1}+\frac{n}{2}-2-\frac{3}{n+1}\right) \qquad (n \geqslant 2). \tag{37}$$

5. After these preliminaries we begin with the first of the equations (17) which, keeping the constants of integration apart and assuming $n \geqslant 3$, we write in the form

$$a_n = 2(\alpha_0\alpha_n + \alpha_1\alpha_{n-1} + \gamma_0\gamma_n + \gamma_1\gamma_{n-1}) + \sum_{\nu=2}^{n-2}(\alpha_\nu\alpha_{n-\nu} + \gamma_\nu\gamma_{n-\nu}) \tag{38}$$

where the sum is interpreted as zero, if $n = 3$.

We now assume that for a certain $n \geqslant 3$ and for $2 \leq \nu \leq n$ is

$$|\alpha_\nu| \leq \alpha H_\nu,\ |\beta_\nu| \leq \beta H_\nu,\ |\gamma_\nu| \leq \gamma H_\nu,\ |\delta_\nu| \leq \delta H_\nu. \tag{39}$$

In that case we get from (38)

$$|a_n| \leq 2(\alpha|\alpha_0| + \gamma|\gamma_0|)H_n + 2(\alpha|\alpha_1| + \gamma|\gamma_1|)H_{n-1} + (\alpha^2+\gamma^2)\sum_{\nu=2}^{n-2} H_\nu H_{n-\nu}. \tag{40}$$

By (35) we obtain from this

$$\left.\begin{aligned}|a_n| \leq 2\,(\alpha\,|\alpha_0| + \gamma\,|\gamma_0|)\frac{\lambda^n}{(n+2)^{(2)}} + 2\,(\alpha\,|\alpha_1| + \gamma\,|\gamma_1|)\frac{\lambda^{n-1}}{(n+1)^{(2)}}\\ + \frac{2}{3}(\alpha^2+\gamma^2)\left(6\,s_n + n - 10 - \frac{12}{n}\right)\frac{\lambda^n}{(n+4)^{(3)}}.\end{aligned}\right\} \quad (41)$$

In this, the last term is left out for $n < 4$, but since it vanishes for $n = 3$, (41) is valid for $n \geqslant 3$.

A sufficient condition for $|a_n| \leq AH_n$ for $n \geqslant 3$ is therefore that the right-hand side of (41) is $\leq A\frac{\lambda^n}{(n+2)^{(2)}}$ which, after multiplication by $\frac{1}{2}(n+2)^{(2)}\lambda^{-n}$ may be written

$$\alpha\,|\alpha_0| + \gamma\,|\gamma_0| + (\alpha\,|\alpha_1| + \gamma\,|\gamma_1|)\frac{n+2}{n\lambda} + \frac{\alpha^2+\gamma^2}{3}\left(6\,s_n + n - 10 - \frac{12}{n}\right)\frac{n+1}{(n+4)^{(2)}} \leq \frac{A}{2}. \quad (42)$$

From (42) we derive a sufficient condition which is independent of n, replacing the factors depending on n by absolute numbers which are at least as large. We first have

$$\frac{n+2}{n} = 1 + \frac{2}{n} \leq \frac{5}{3} \qquad (n \geqslant 3) \quad (43)$$

and proceed to prove that

$$\left(6\,s_n + n - 10 - \frac{12}{n}\right)\frac{n+1}{(n+4)^{(2)}} < 2 \qquad (n \geqslant 3). \quad (44)$$

Now it is verified directly that (44) is valid for $n = 3$ and $n = 4$, so that in the remainder of the proof we may assume $n \geqslant 5$. But we have obviously

$$s_n \leq s_k + \frac{n-k}{k+1} \qquad (n \geqslant k) \quad (45)$$

whence, in particular,

$$s_n \leq s_5 + \frac{n-5}{6} = \frac{n}{6} + \frac{29}{20} \qquad (n \geqslant 5) \quad (46)$$

and inserting this in (44) we get the more rigid inequality

$$\frac{2n - 1\cdot 3 - \dfrac{12}{n}}{n+4}\cdot\frac{n+1}{n+3} < 2$$

which is obvious, the first factor on the left being less than 2, and the second less than 1.

By (43) and (44) we finally obtain from (42) the following sufficient condition, which does not depend on n, for $|a_n| \leq AH_n$

$$\alpha\,|\,\alpha_0\,| + \gamma\,|\,\gamma_0\,| + \frac{5}{3\lambda}(\alpha\,|\,\alpha_1\,| + \gamma\,|\,\gamma_1\,|) + \frac{2}{3}(\alpha^2 + \gamma^2) \leq \frac{A}{2}, \tag{47}$$

always provided that $n \geqslant 3$.

After this, a comparison of the two first equations (17) shows that we obtain from (47), by a simple exchange of letters, as a sufficient condition for $|b_n| \leq BH_n$ for $n \geqslant 3$

$$\beta\,|\,\beta_0\,| + \delta\,|\,\delta_0\,| + \frac{5}{3\lambda}(\beta\,|\,\beta_1\,| + \delta\,|\,\delta_1\,|) + \frac{2}{3}(\beta^2 + \delta^2) \leq \frac{B}{2}. \tag{48}$$

As regards the third equation (17) we begin by writing it in the form, valid for $n \geqslant 3$,

$$\left.\begin{aligned} c_n = a_n + b_n + 2\,(\alpha_0\beta_n + \alpha_1\beta_{n-1} + \beta_0\alpha_n + \beta_1\alpha_{n-1} + \gamma_0\delta_n + \gamma_1\delta_{n-1} + \delta_0\gamma_n + \delta_1\gamma_{n-1}) \\ + 2\sum_{\nu=2}^{n-2}(\alpha_\nu\beta_{n-\nu} + \gamma_\nu\delta_{n-\nu}). \end{aligned}\right\} \tag{49}$$

From this we obtain in the same way as above

$$\left.\begin{aligned} |\,c_n\,| \leq 2\,H_n\left(\frac{A+B}{2} + \alpha\,|\,\beta_0\,| + \beta\,|\,\alpha_0\,| + \gamma\,|\,\delta_0\,| + \delta\,|\,\gamma_0\,|\,\right) \\ + 2\,H_{n-1}(\alpha\,|\,\beta_1\,| + \beta\,|\,\alpha_1\,| + \gamma\,|\,\delta_1\,| + \delta\,|\,\gamma_1\,|) + 2\,(\alpha\beta + \gamma\delta)\sum_{\nu=2}^{n-2} H_\nu H_{n-\nu} \end{aligned}\right\} \tag{50}$$

A sufficient condition for $|c_n| \leq CH_n$ is therefore that the right-hand side of (50) is $\leq C\dfrac{\lambda^n}{(n+2)^{(2)}}$, and this may, by (30) and (35) and after multiplication by $\frac{1}{2}(n+2)^{(2)}\lambda^{-n}$, be written

$$\left.\begin{aligned} \frac{A+B}{2} + \alpha\,|\,\beta_0\,| + \beta\,|\,\alpha_0\,| + \gamma\,|\,\delta_0\,| + \delta\,|\,\gamma_0\,| + (\alpha\,|\,\beta_1\,| + \beta\,|\,\alpha_1\,| + \gamma\,|\,\delta_1\,| + \delta\,|\,\gamma_1\,|)\,\frac{n+2}{n\,\lambda} \\ + \frac{2}{3}(\alpha\beta + \gamma\delta)\left(6\,s_n + n - 10 - \frac{12}{n}\right)\frac{n+1}{(n+4)^{(2)}} \leq \frac{C}{2}. \end{aligned}\right\} \tag{51}$$

By (43) and (44) we obtain finally the more severe, but of n independent, condition, valid for $n \geqslant 3$, for $|c_n| \leq CH_n$

$$\left.\begin{array}{r}\alpha\,|\,\beta_0\,|+\beta\,|\,\alpha_0\,|+\gamma\,|\,\delta_0\,|+\delta\,|\,\gamma_0\,|+\dfrac{5}{3\lambda}(\alpha\,|\,\beta_1\,|+\beta\,|\,\alpha_1\,|+\gamma\,|\,\delta_1\,|+\delta\,|\,\gamma_1\,|)\\ +\dfrac{4}{3}(\alpha\beta+\gamma\delta)\leq\dfrac{1}{2}(C-A-B).\end{array}\right\}\quad(52)$$

6. We now consider (18), assuming that for a certain $n \geqslant 2$ we have proved that for $1 \leq \nu \leq n$

$$|\,a_\nu\,| \leq A H_\nu, \qquad |\,b_\nu\,| \leq B H_\nu, \qquad |\,c_\nu\,| \leq C H_\nu \tag{53}$$

and for $0 \leq \nu \leq n-1$ that

$$|\,d_\nu\,| \leq D H_\nu, \qquad |\,e_\nu\,| \leq E H_\nu, \qquad |\,f_\nu\,| \leq F H_\nu. \tag{54}$$

We then obtain from the first of the equations (18)

$$2\,n a_0\,|\,d_n\,| \leq D A\left(3\,n\sum_{\nu=0}^{n-1} H_\nu H_{n-\nu} + \sum_{\nu=1}^{n-1} \nu H_\nu H_{n-\nu}\right) \tag{55}$$

whence by (34) and (37), after reduction

$$2\,a_0\,|\,d_n\,| \leq D A\left(14\,s_{n+1} + 5\,n - 5 - \frac{12}{n+1}\right)\frac{\lambda^n}{(n+4)^{(3)}}. \tag{56}$$

A sufficient condition for $|\,d_n\,| \leq D H_n$ is therefore that the right-hand side of (56) is $\leq 2\,a_0 D \dfrac{\lambda^n}{(n+2)^{(2)}}$, which may be written

$$\left(14\,s_{n+1} + 5\,n - 5 - \frac{12}{n+1}\right)\frac{n+1}{(n+4)^{(2)}} \leq \frac{2\,a_0}{A}. \tag{57}$$

We will now show that this condition may be replaced by the more restricted sufficient condition

$$3\,A \leq a_0 \tag{58}$$

which is independent of n. This comes to proving that

$$\left(14\,s_{n+1} + 5\,n - 5 - \frac{12}{n+1}\right)\frac{n+1}{(n+4)^{(2)}} \leq 6 \tag{59}$$

or

$$s_{n+1} \leq \frac{n-1}{14} + 3 + \frac{24}{7\,(n+1)}. \tag{60}$$

Now it is seen by a table of s_n[1] that (60) is satisfied for $n < 12$, while for $n \geqslant 12$ we may employ

$$s_{n+1} \leq s_{13} + \frac{n-12}{14}$$

which inserted in (60) gives, after reduction, the more rigid condition

$$s_{13} < \frac{53}{14} + \frac{24}{7(n+1)}$$

which is also satisfied. Hence, (60) and therefore (58) are proved.

Since the second and third of the equations (18) are obtained from the first by a simple exchange of letters, we may now write down as a sufficient condition for the validity of (54) for $0 \leq \nu \leq n$

$$3A \leq a_0, \qquad 3B \leq b_0, \qquad 3C \leq c_0. \tag{61}$$

7. As regards finally (16), we isolate the constants of integration, assume $n \geqslant 2$ and write the first of these equations in the form

$$\left.\begin{aligned}(n+2)^{(2)}\alpha_{n+2} = m_1(\beta_0 e_n + \beta_1 e_{n-1} - \varepsilon_0 f_n - \varepsilon_1 f_{n-1}) - M_1(\alpha_0 d_n + \alpha_1 d_{n-1})\\ + m_1 \sum_{\nu=2}^{n}(\beta_\nu e_{n-\nu} - \varepsilon_\nu f_{n-\nu}) - M_1 \sum_{\nu=2}^{n} \alpha_\nu d_{n-\nu}.\end{aligned}\right\} \tag{62}$$

We write for abbreviation

$$P_1 = m_1(F+E), P_2 = m_2(F+D), Q_1 = m_1 F + M_1 D, Q_2 = m_2 F + M_2 E \tag{63}$$

and assume that (54) is satisfied for $0 \leq \nu \leq n$, (39) for $2 \leq \nu \leq n$. From (32) we obtain

$$\sum_{\nu=2}^{n} H_\nu H_{n-\nu} = \frac{\lambda^n}{(n+4)^{(3)}}\left(4 s_{n-1} + \frac{4n-7}{3} + \frac{2}{n+1}\right) \quad (n \geqslant 2) \tag{64}$$

and thereafter from (62) for $n \geqslant 2$

$$\left.\begin{aligned}(n+2)^{(2)} |\alpha_{n+2}| \leq (|\alpha_0| Q_1 + |\beta_0| P_1)\frac{\lambda^n}{(n+2)^{(2)}} + (|\alpha_1| Q_1 + |\beta_1| P_1)\frac{\lambda^{n-1}}{(n+1)^{(2)}}\\ + (\alpha Q_1 + \beta P_1)\left(4 s_{n-1} + \frac{4n-7}{3} + \frac{2}{n+1}\right)\frac{\lambda^n}{(n+4)^{(3)}}.\end{aligned}\right\} \tag{65}$$

[1] See, for instance, J. W. Glover: Tables of Applied Mathematics, Ann Arbor, Michigan, 1923, p. 456.

A sufficient condition for $|\alpha_{n+2}| \leq \alpha H_{n+2}$ is therefore that the right-hand side of (65) is $\leq (n+2)^{(2)} \alpha H_{n+2}$, which after multiplication by $\lambda^{-n} \frac{(n+4)^{(2)}}{(n+2)^{(2)}}$ may be written

$$\left.\begin{aligned}(|\alpha_0| Q_1 + |\beta_0| P_1) \frac{(n+4)(n+3)}{(n+2)^2(n+1)^2} + (|\alpha_1| Q_1 + |\beta_1| P_1) \frac{(n+4)(n+3)\lambda^{-1}}{(n+2)(n+1)^2 n} \\ + \frac{\alpha Q_1 + \beta P_1}{(n+2)^2(n+1)} \left(4 s_{n-1} + \frac{4n-7}{3} + \frac{2}{n+1}\right) \leq \alpha \lambda^2.\end{aligned}\right\} \quad (66)$$

In order to find a sufficient condition that does not depend on n we observe that, since we have assumed $n \geqslant 2$,

$$\frac{(n+4)(n+3)}{(n+2)^2(n+1)^2} = \left(1 + \frac{2}{n+2}\right)\left(1 + \frac{1}{n+2}\right)\frac{1}{(n+1)^2} \leq \frac{5}{24} \quad (67)$$

and

$$\frac{(n+4)(n+3)}{(n+2)(n+1)^2 n} = \left(1 + \frac{2}{n+2}\right)\left(1 + \frac{3}{n}\right)\frac{1}{(n+1)^2} \leq \frac{5}{12}. \quad (68)$$

We will finally show that for $n \geqslant 2$

$$\frac{1}{(n+2)^2(n+1)}\left(4 s_{n-1} + \frac{4n-7}{3} + \frac{2}{n+1}\right) \leq \frac{5}{48}. \quad (69)$$

For $n = 2$ it is seen directly that this holds. For $n \geqslant 3$ we insert the inequality

$$s_{n-1} \leq \frac{1}{2} + \frac{n}{3} \qquad (n \geqslant 3) \quad (70)$$

resulting from (45) for $k = 2$. The result may be written

$$128 \leq 5(n+2)^2 + 48 \frac{3n+1}{(n+1)^2} \qquad (n \geqslant 3)$$

which is easily verified. Hence (69) is proved for $n \geqslant 2$.

If now we insert (67) – (69) in (66), we obtain the sufficient condition, valid for $n \geqslant 2$, but otherwise independent of n,

$$2(|\alpha_0| Q_1 + |\beta_0| P_1) + \frac{4}{\lambda}(|\alpha_1| Q_1 + |\beta_1| P_1) + \alpha Q_1 + \beta P_1 \leq \frac{48}{5} \alpha \lambda^2. \quad (71)$$

Since the three last equations (16) are obtained from the first by a simple exchange of letters, we may now by (71) write down the following sufficient conditions, valid for $n \geqslant 2$

$$2(|\alpha_0|P_2+|\beta_0|Q_2)+\frac{4}{\lambda}(|\alpha_1|P_2+|\beta_1|Q_2)+\alpha P_2+\beta Q_2 \leq \frac{48}{5}\beta\lambda^2. \quad (72)$$

$$2(|\gamma_0|Q_1+|\delta_0|P_1)+\frac{4}{\lambda}(|\gamma_1|Q_1+|\delta_1|P_1)+\gamma Q_1+\delta P_1 \leq \frac{48}{5}\gamma\lambda^2. \quad (73)$$

$$2(|\gamma_0|P_2+|\delta_0|Q_2)+\frac{4}{\lambda}(|\gamma_1|P_2+|\delta_1|Q_2)+\gamma P_2+\delta Q_2 \leq \frac{48}{5}\delta\lambda^2. \quad (74)$$

8. We may summarize the result of the preceding investigation thus: If (39) is satisfied for $2 \leq \nu \leq 3$, (53) for $1 \leq \nu \leq 2$, (54) for $0 \leq \nu \leq 2$, and if, besides, all the inequalities (47), (48), (52), (61), (71)–(74) are satisfied, then (12) and (13) are convergent provided that $\Sigma H_\nu t^\nu$ converges, that is, for $|t| \leq \frac{1}{\lambda}$.

It may be observed that the condition (52) implies that $A+B<C$.

The question arises whether it is always possible, when the initial values (19) are arbitrarily given, to find such values of λ, α, β, γ, δ, A, B, C, D, E, F that the aforesaid inequalities are all satisfied. This question must be answered in the affirmative. To begin with, λ can always be chosen so large that (71)–(74) are satisfied and that (47), (48) and (52) are reduced to

$$\alpha|\alpha_0|+\gamma|\gamma_0|+\frac{2}{3}(\alpha^2+\gamma^2)<\frac{A}{2}$$

$$\beta|\beta_0|+\delta|\delta_0|+\frac{2}{3}(\beta^2+\delta^2)<\frac{B}{2}$$

$$\alpha|\beta_0|+\beta|\alpha_0|+\gamma|\delta_0|+\delta|\gamma_0|+\frac{4}{3}(\alpha\beta+\gamma\delta)<\frac{1}{2}(C-A-B)$$

while (61) is unchanged. We now choose A, B and C so small that (61) is satisfied and, besides, $A+B<C$. After this α, β, γ, δ may be chosen so small that the three reduced inequalities are satisfied. Small values of A, B, C, α, β, γ, δ can always be compensated by an increase of λ.

9. As a simple numerical example to show the practical working of the recurrence formulas we choose

$$m_1 = 1, m_2 = 2, m_3 = 3 \quad (75)$$

so that

$$M_1 = 5, M_2 = 4, M_3 = 3, \quad (76)$$

and for the initial values

$$\left.\begin{aligned} &\alpha_0 = \cdot 5, \quad \beta_0 = \cdot 9, \quad \gamma_0 = 1 \cdot 2, \quad \delta_0 = -1 \cdot 2 \\ &\alpha_1 = \cdot 15, \quad \beta_1 = -\cdot 1, \quad \gamma_1 = -\cdot 2, \quad \delta_1 = -\cdot 3. \end{aligned}\right\} \tag{77}$$

From these I derive by (20) – (29) the coefficients in the table below[1] where the exact values of d_0, e_0, f_0 are

$$d_0 = \frac{1}{2 \cdot 197}, \quad e_0 = \frac{1}{3 \cdot 375}, \quad f_0 = \frac{1}{2 \cdot 744} \tag{78}$$

and where at the time $t = 0$

$$r_1 = \sqrt{a_0} = 1 \cdot 3, \quad r_2 = \sqrt{b_0} = 1 \cdot 5, \quad r_3 = \sqrt{c_0} = 1 \cdot 4. \tag{79}$$

As regards the convergence, the sufficient conditions established above are satisfied if we choose, for instance

$$\left.\begin{aligned} &\lambda = 20, \quad \alpha = \cdot 021, \quad \beta = \cdot 025, \quad \gamma = \cdot 047, \quad \delta = \cdot 038, \\ &A = \cdot 14, \quad B = \cdot 17, \quad C = \cdot 60, \quad D = \cdot 92, \quad E = \cdot 60, \quad F = \cdot 73. \end{aligned}\right\} \tag{80}$$

The expansions (12) and (13) are therefore at least convergent for $|t| \leq \frac{1}{20}$.

I find for $t = \frac{1}{20}$

$$\left.\begin{aligned} &\varrho_1 = 1 \cdot 66270 \qquad \sigma_1 = \cdot 466385 \\ &\varrho_2 = 2 \cdot 26598 \qquad \sigma_2 = \cdot 293155 \\ &\varrho_3 = 1 \cdot 95711 \qquad \sigma_3 = \cdot 365223 \end{aligned}\right\} \tag{81}$$

and from $r_i = \sqrt{\varrho_i}$

$$r_1 = 1 \cdot 28946, \quad r_2 = 1 \cdot 50532, \quad r_3 = 1 \cdot 39897. \tag{82}$$

10. A considerable simplification is obtained in the particular case where

$$\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = 0. \tag{83}$$

Under these circumstances there are only the four arbitrary constants α_0, β_0, γ_0, δ_0 left. The significance of (83) is that at the outset we have

$$\frac{d\xi_1}{dt} = \frac{d\xi_2}{dt} = 0, \quad \frac{d\eta_1}{dt} = \frac{d\eta_2}{dt} = 0 \qquad (t = 0) \tag{84}$$

[1] The number of decimals retained in the table has been cut down to seven.

TABLE.

ν	α_ν	β_ν	γ_ν	δ_ν	ε_ν	ζ_ν
0	·5	·9	1·2	—1·2	1·4	·0
1	·15	—·1	— ·2	— ·3	·05	— ·5
2	— ·6907264	— ·8159544	—1·5432762	1·2573105	—1·5066808	— ·2859657
3	— ·1273093	·1408788	— ·0205691	·0576473	·0135695	·0370782

ν	a_ν	b_ν	c_ν	d_ν	e_ν	f_ν
0	1·69	2·25	1·96	·4551661	·2962963	·3644315
1	— ·33	·54	·14	·1333179	— ·1066667	— ·0390462
2	—4·3320893	—4·3862630	—3·9662061	1·7826770	·8984223	1·1096677
3	·2334175	— ·4759671	·1732921	·7674704	— ·4347952	— ·2461544

or, expressed by the coordinates in the absolute movement,

$$\frac{dx_1}{dt} = \frac{dx_2}{dt} = \frac{dx_3}{dt}, \quad \frac{dy_1}{dt} = \frac{dy_2}{dt} = \frac{dy_3}{dt} \qquad (t = 0). \tag{85}$$

From (83) follows at once by (22) and (23) that

$$a_1 = b_1 = c_1 = d_1 = e_1 = f_1 = 0, \tag{86}$$

whereafter the general recurrence formulas (16)–(18) show that all the coefficients of the odd order vanish.

11. We shall finally call attention to another particular case where considerable simplifications occur. Let h be an arbitrary constant, and let us for $\nu = 0$ and $\nu = 1$ choose

$$\gamma_\nu = h\alpha_\nu, \quad \delta_\nu = h\beta_\nu, \quad \text{whence } \zeta_\nu = h\varepsilon_\nu. \tag{87}$$

In that case comparison between the first and third, and between the second and fourth, of the equations (16) shows that (87) is valid for all ν. It follows that $(\alpha\delta)_n = (\beta\gamma)_n$, so that

$$\xi_1\eta_2 = \eta_1\xi_2 \tag{88}$$

or

$$\frac{y_3 - y_2}{x_3 - x_2} = \frac{y_3 - y_1}{x_3 - x_1}. \tag{89}$$

But a simple geometrical consideration shows that this means that the three bodies are always situated on a straight line.

Under these circumstances the calculation of the coefficients is simplified, because (87) shows that the two last equations (16) are identical with the two first and can be left out, while (17) is reduced to

$$\left.\begin{aligned} a_n &= (1+h^2)\,(\alpha\alpha)_n \\ b_n &= (1+h^2)\,(\beta\beta)_n \\ c_n &= a_n + b_n + 2\,(1+h^2)\,(\alpha\beta)_n . \end{aligned}\right\} \tag{90}$$

12. It was mentioned at the outset that if the absolute positions in the plane of the three bodies are required they can be determined afterwards. We will briefly indicate how this may be done.

Writing the third of the equations (6) in the form

$$\frac{d^2 x_3}{dt^2} = m_2 \xi_1 \sigma_1 - m_1 \xi_2 \sigma_2 \tag{91}$$

and putting

$$u_n = m_2\,(\alpha d)_n - m_1\,(\beta e)_n \tag{92}$$

we have

$$\frac{d^2 x_3}{dt^2} = \sum_{n=0}^{\infty} u_n t^n , \tag{93}$$

and integrating this twice, introducing thus two more arbitrary constants, we have the expansion of x_3, whereafter by (7)

$$x_2 = x_3 + \xi_1 , \qquad x_1 = x_3 - \xi_2 . \tag{94}$$

The equation for y_3

$$\frac{d^2 y_3}{dt^2} = m_2 \eta_1 \sigma_1 - m_1 \eta_2 \sigma_2 \tag{95}$$

may be treated in the same way. Putting

$$v_n = m_2\,(\gamma d)_n - m_1\,(\delta e)_n \tag{96}$$

we have

$$\frac{d^2 y_3}{dt^2} = \sum_{n=0}^{\infty} v_n t^n \tag{97}$$

whence, introducing two more arbitrary constants, we find y_3 and finally by (7)

$$y_2 = y_3 + \eta_1, \qquad y_1 = y_3 - \eta_2. \tag{98}$$

If the values at $t = 0$ of x_i, $\frac{dx_i}{dt}$, y_i, $\frac{dy_i}{dt}$ are chosen arbitrarily, the corresponding values of (19), or ξ_i, $\frac{d\xi_i}{dt}$, η_i, $\frac{d\eta_i}{dt}$ at $t = 0$, result immediately from (7).

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