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ON THE DUAL SPACES OF THE BESICOVITCH ALMOST PERIODIC SPACES

BY

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Introduction.

As is well known, see for instance [5], p. 36, for every p, $1 \leq p < \infty$, a B^p-a. p. function f is a function to which there exists a sequence of ordinary almost periodic functions φ_n such that

$$\| f - \varphi_n \|_{B^p} \to 0 \text{ for } n \to \infty.$$

Here the B^p-norm is defined by

$$\| f \|_{B^p} = \overline{\lim_{T \to \infty}} \left(\frac{1}{2T} \int_{-T}^{T} | f(x) |^p d x \right)^{\frac{1}{p}} = \left(\overline{M} \{ | f(x) |^p \} \right)^{\frac{1}{p}}.$$

If in particular $\| f \|_{B^p} = 0$, the function f is called a B^p-zero function. If we set $f \equiv g$ when $\| f - g \|_{B^p} = 0$, the equivalence classes for this relation are called B^p-a. p. points. Multiplication of a B^p-a. p. point by a complex constant, addition of two B^p-a. p. points, and the B^p-norm of a B^p-a. p. point are defined in the natural way. Thus the set of B^p-a. p. points becomes a linear metric space, [5], pp. 37—39. Since the B^p-a. p. space is complete, see for instance [5], pp. 54—57, it is a Banach space.

To every B^p-a. p. function f, $1 \leq p < \infty$, is associated a Fourier series

$$f(x) \sim \sum a(\lambda) e^{i \lambda x},$$

where the coefficient function

$$a(\lambda) = M \{ f(x) e^{-i \lambda x} \}$$

is $\neq 0$ only for a denumerable number of values λ, the so-called Fourier exponents of f, [2], p. 262. Since all functions in a B^p-a. p. point have the same Fourier series, this Fourier series is called the Fourier series of the B^p-a. p. point.

Let **M** be an arbitrary module of real numbers, i. e., a set of real numbers which together with λ_1 and λ_2 also contains $\lambda_1 - \lambda_2$. A B^p-a. p. function (B^p-a. p. point) is called a B^p-a. p.—**M** function (B^p-a. p.—**M** point) if all its Fourier exponents belong to **M**. The subspace of the B^p-a. p. space consisting of all B^p-a. p.—**M** points is a linear closed subspace and hence a Banach space. If in particular **M** is the module of all real numbers, the B^p-a. p.—**M** space is the B^p-a. p. space itself.

We consider a complex bounded linear functional A on the space of B^p-a. p.—**M** *functions*, i. e., a complex functional which satisfies

$$A(\lambda f) = \lambda A f \qquad (\lambda \text{ complex})$$

$$A(f+g) = Af + Ag$$

$$|Af| \leq C \, \| f \|_{B^p},$$

where C is independent of f. Here we may assume C chosen as the smallest of its possible values. C is then called the norm of A and denoted by $\| A \|$. It is obvious that A takes the same value on all B^p-a. p.—**M** functions in a B^p-a. p.—**M** point. Thus A may also be considered as a bounded linear functional on the B^p-a. p.—**M** space. With usual addition, usual multiplication by a complex constant, and the above norm, the set of bounded linear functionals on the B^p-a. p.—**M** space is a Banach space, the so-called dual space of the B^p-a. p.—**M** space.

We shall prove in the present paper that for $1 < p < \infty$ and any module **M** of real numbers the dual space of the B^p-a. p.—**M** space is the B^q-a. p.—**M** space where q is determined by $1/p + 1/q = 1$. The isomorphism (i. e., the linear one-to-one isometric mapping) of the dual space of the B^p-a. p.—**M** space on the B^q-a. p.—**M** space is given by

$$A_g \rightarrow g \quad \text{where} \quad A_g f = M\{f\bar{g}\}.$$

The dual space of the B^1-a. p.—**M** space will also be completely characterized.

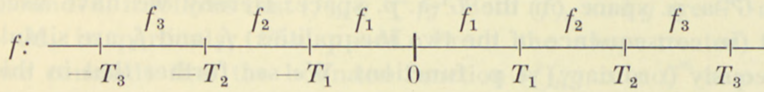
We shall deduce this main result by two rather different methods. The first method is the most elementary one and uses only the ordinary theory of generalized almost periodic functions. It is based on previous results by R. Doss [7], [8], and is an extension of the method used by Doss. This method is set forth in Part I of the paper.

The second method, which is set forth in Part II of the paper, consists in the establishment of a close correspondance between the B^p-a. p.—**M** points and the measurable p-integrable functions on the Bohr compactification of the real axis by all ordinary a. p.—**M** functions. When this correspondence is established, our main result concerning the dual space of the B^p-a. p.—**M** space, $1 \leq p < \infty$, is an immediate consequence of the generalization to the abstract case of F. Riesz's classical result concerning the dual space of the space L^p of measurable p-integrable functions, $1 \leq p < \infty$.

Part I.

1. Preparations.

1. We have mentioned in the Introduction that the B^p-a. p. space, $1 \leq p < \infty$, is a complete space; or, in other words, that if f_n is a B^p-fundamental sequence of B^p-a. p. functions: $\| f_m - f_n \|_{B^p} \to 0$ for $m, n \to \infty$, then there exists a B^p-a. p. function f such that $\| f - f_n \|_{B^p} \to 0$ for $n \to \infty$. We shall use moreover that f, as shown in [5], pp. 54—57, can be constructed "from pieces of the f_n" as indicated on the following figure



where $0 = T_0 < T_1 < T_2 < \cdots \to \infty$ and the only extra demand to T_n is of the form

$$T_n > t\,(T_0, T_1, \cdots, T_{n-1}),\; n = 1, 2, \cdots.$$

When in the following the letter G is applied (instead of the usual B), this indicates that the theorems are true for all three types of generalized almost periodic functions, the Stepanoff a. p. functions, the Weyl a. p. functions, and the Besicovitch a. p. functions (for their definitions see for instance [5], pp. 33—39).

2. On account of Hölder's inequality we have $\| f \|_{G^{p_1}} \leq \| f \|_{G^{p_2}}$ for $1 \leq p_1 < p_2$. Hence a G^{p_2}-a. p. function (G^{p_2}-zero function) is also a G^{p_1}-a. p. function (G^{p_1}-zero function).

3. A bounded G^1-a. p. function (bounded G^1-zero function) is a G^p-a. p. function (G^p-zero function) for all p, $1 \leq p < \infty$; [5], p. 62. We shall call such a bounded G^1-a. p. function (bounded G^1-zero function) a G^∞-a. p. function (G^∞-zero function).

4. Deeper-lying theorem: *A B^1-a. p. point which contains a B^P-bounded function for a fixed P, $1 < P < \infty$, contains also a B^P-a. p. function.* [5], pp. 99—106.

5. We consider the inequalities

$$\left| \, |f|^{\frac{1}{p}} \operatorname{sign} f - |\varphi|^{\frac{1}{p}} \operatorname{sign} \varphi \, \right| \leq 2^p \, | f - \varphi |$$

and

$$\left| \, |f|^p \operatorname{sign} f - |\varphi|^p \operatorname{sign} \varphi \, \right| \leq 2p \, | f - \varphi | \left(|f| + |\varphi| \right)^{p-1},$$

where $1 \leq p < \infty$. See [6], pp. 220—221, exercise 10. These inequalities, the latter in connection with Hölder's inequality, show that the mapping

$$(1) \qquad f_1 \to f_2,$$

where $f_2 = |f_1|^p \operatorname{sign} f_1$ and hence $f_1 = |f_2|^{\frac{1}{p}} \operatorname{sign} f_2$ is (or more correctly: may be considered as) a homeomorphic mapping of the G^p-a. p. space on the G^1-a. p. space. Hereby we have used that (in consequence of the two inequalities) f_1 and f_2 are simultaneously (ordinary) a. p. functions. We see further that in this case they "majorize" each other; [4], p. 60. Hence (l. c.) f_1 and f_2 are simultaneously a. p.—**M** functions. Since a G^p-a. p.—**M** function is a function which can be G^p-approximated by a. p.—**M** functions (cf. 8. below), we conclude that if f_1 is a G^p-a. p.—**M** function, then f_2 is a G^1-a. p.—**M** function, and conversely. Hence (1) is also a homeomorphic mapping of the G^p-a. p.—**M** space on the G^1-a. p.—**M** space.

Combining this result and the corresponding result with p replaced by q, $1 \leq q < \infty$, we see finally that

$$f_1 \to f_2,$$

where $f_2 = |f_1|^{\frac{p}{q}} \operatorname{sign} f_1$, and hence $f_1 = |f_2|^{\frac{q}{p}} \operatorname{sign} f_2$ is a homeomorphic mapping of the G^p-a. p.—**M** space on the G^q-a. p.—**M** space for any module **M** of real numbers. In particular, when f_1 is G^p-a. p.—**M**, then f_2 is G^q-a. p.—**M**, and conversely. Cf. [11], pp. 422—423.

6. If φ_n is a sequence of $G^{\alpha p}$-a. p. functions which $G^{\alpha p}$-converges to the $G^{\alpha p}$-a. p. function f and ψ_n is a sequence of $G^{\alpha q}$-a. p. functions which $G^{\alpha q}$-converges to the $G^{\alpha q}$-a. p. function g, for fixed $\alpha \geqq 1$, $1 < p < \infty$, $1 < q < \infty$, $1/p + 1/q = 1$, then $\varphi_n \psi_n$ will G^{α}-converge to fg. This follows easily by application of Hölder's and Minkowski's inequalities. In particular, when f is a $G^{\alpha p}$-a. p.—**M** function and g is a $G^{\alpha q}$-a. p.—**M** function, then fg is a G^{α}-a. p.—**M** function. Cf. [11], pp. 416—417.

If f is a G^{α}-a. p.—**M** function and g is a G^{∞}-a. p.—**M** function for fixed $\alpha \geqq 1$, then fg is a G^{α}-a. p.—**M** function. In order to see this we introduce the cut-off function

$$(f(x))_n = \begin{cases} f(x) & \text{for } |f(x)| \leqq n \\ n \operatorname{sign} f(x) & \text{for } |f(x)| \geqq n. \end{cases}$$

Since

$$|(f)_n - (\varphi)_n| \leqq |f - \varphi|$$

we see that if f is a. p.—**M**, then $(f)_n$ is a. p.—**M** as it is majorized by f; [4], p. 60. It follows that in the general case $(f)_n$ is G^{∞}-a. p.—**M**. Further $(f)_n \overset{G^{\alpha}}{\to} f$ for $n \to \infty$; [5], pp. 44—45. Since g is bounded, it follows that $(f)_n\, g \overset{G^{\alpha}}{\to} fg$. Both $(f)_n$ and g are G^{∞}-a. p. —**M**, in particular G^2-a. p.—**M**. From the above-treated case we conclude that the bounded function $(f)_n g$ is a G^{∞}-a. p.—**M** function. Then the G^{α}-limit fg is a G^{α}-a. p.—**M** function, as was to be proved.

7. Let **M** be a denumerable module of real numbers α_1, $\alpha_2, \cdots$. By a sequence of Bochner—Fejér kernels belonging to **M** we understand a sequence of non-negative trigonometric polynomials with exponents from **M**, positive coefficients, mean values 1, and which converge formally to $\sum e^{i\alpha_n x}$. It follows that the coefficients are $\leqq 1$.

Let

$$\sum a(\lambda_n)\, e^{i\lambda_n x} = \sum a(\lambda)\, e^{i\lambda x}$$

be a trigonometric series and **M** an arbitrary denumerable module of real numbers $\alpha_1, \alpha_2, \cdots$. Let

$$k_m(x) = \sum d_n^{(m)} e^{i\alpha_n x}$$

be a sequence of Bochner—Fejér kernels belonging to **M**. Then

$$\sigma_m(x) = \sum d_n^{(m)} a(\alpha_n) e^{i\alpha_n x}$$

is called a Bochner—Fejér sequence belonging to **M** of the trigonometric series. If every λ for which $a(\lambda) \neq 0$, belongs to **M**, then the series is said to belong to **M**, and σ_m is called a *full* Bochner—Fejér sequence of the series.

8. Let f be a G^p-a. p. function for a fixed p, $1 \leq p < \infty$, with the Fourier series

$$f(x) \sim \sum a(\lambda) e^{i\lambda x}.$$

Let **M** be a denumerable module and σ_m a Bochner—Fejér sequence belonging to **M** of the Fourier series. Then $\|\sigma_m\|_{G^p} \leq \|f\|_{G^p}$ (see [2], pp. 263—266). If $|f(x)| \leq C$ for all x, then $|\sigma_m(x)| \leq C$ for all x. If σ_m is a full Bochner-Fejér sequence of the series, then $\sigma_m \overset{G^p}{\to} f$ for $m \to \infty$; [2], pp. 262—266.

9. If $f \sim \sum a(\lambda) e^{i\lambda x}$ is a B^1-a. p. function and $g(x) = \sum b(\lambda) e^{i\lambda x}$ is a trigonometric polynomial, then (obviously)

$$M\{f\bar{g}\} = \sum a(\lambda) \overline{b(\lambda)}.$$

If $f \sim \sum a(\lambda) e^{i\lambda x}$ is a B^p-a. p. function and $g \sim \sum b(\lambda) e^{i\lambda x}$ is a B^q-a. p. function for fixed p and q, $1 \leq p < \infty$, $1 < q \leq \infty$, $1/p + 1/q = 1$, and furthermore $a(\lambda)\overline{b(\lambda)} = 0$ for all λ, then

$$M\{f\bar{g}\} = 0.$$

In order to see this, let σ_m be a full Bochner-Fejér sequence of f. Then by 8. we have $\sigma_m \overset{B^p}{\to} f$ and, using also the above remark, $M\{\sigma_m \bar{g}\} = 0$. As a result of 6. the function $f\bar{g}$ is B^1-a. p. and $M\{\sigma_m \bar{g}\} \to M\{f\bar{g}\}$. Thus $M\{f\bar{g}\} = 0$, as we had to show.

10. If a sequence A_m of bounded linear functionals on a

Banach space converges weakly, i. e., $A_m f$ converges for every f in the Banach space, then there exists a constant C such that $\|A_m\| \leqq C$ for all m. See for instance [9], p. 21.

2. Necessary and sufficient conditions for a trigonometric series to be the Fourier series of a B^p-a. p. function for a given p, $1 \leqq p \leqq \infty$.

Doss, [7], p. 209, and [8], pp. 89—91, has proved the following two theorems.

Theorem A. *A necessary and sufficient condition for a trigonometric series* $\sum a_n e^{i\lambda_n x}$ *to be the Fourier series of a* B^∞-a. p. *function is that for a full Bochner-Fejér sequence* σ_m *of the series there exists a constant* C *such that* $|\sigma_m(x)| \leqq C$ *for all* x *and all* m.

Theorem B. *A necessary and sufficient condition for a trigonometric series* $\sum a_n e^{i\lambda_n x}$ *to be the Fourier series of a* B^1-a. p. *function is that a full Bochner-Fejér sequence* σ_m *of the series has the following property: To every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $\overline{M}_E\{|\sigma_m(x)|\} < \varepsilon$ *for any measurable set* E *with upper mean-measure* $\overline{m}\, E < \delta$.

The remaining cases, $1 < p < \infty$, are dealt with in the following:

Theorem 1. *Let* p *be a fixed number,* $1 < p < \infty$. *A necessary and sufficient condition for a trigonometric series* $\sum a_n e^{i\lambda_n x}$ *to be the Fourier series of a* B^p-a. p. *function is that for a full Bochner-Fejér sequence* σ_m *of the series there exists a constant* C *such that* $\|\sigma_m\|_{B^p} < C$ *for all* m.

Proof. The necessity of the condition is clear, for if the series is the Fourier series of the B^p-a. p. function f, then as stated in 1, 8. we have $\|\sigma_m\|_{B^p} \leqq \|f\|_{B^p}$.

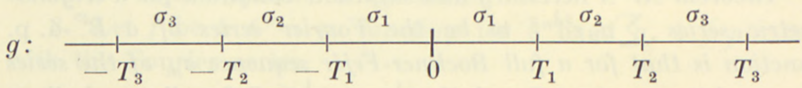
We shall now show that the condition is sufficient. Assuming the condition fulfilled, we show first that the series is the Fourier series of a B^1-a. p. function; and we do this by showing that it fulfils the condition from Theorem B. Let E be an arbitrary measurable set of real numbers; let e denote the characteristic function of E; and let σ_m be a full Bochner-Fejér sequence of the series. Determining q by $1/p + 1/q = 1$ we obtain by Hölder's inequality

$$\overline{M}_E\{|\sigma_m(x)|\} = \overline{M}\{e(x)|\sigma_m(x)|\} \leqq$$

$$(M\{|\sigma_m|^p\})^{\frac{1}{p}}(\overline{M}\{e^q\})^{\frac{1}{q}} = \|\sigma_m\|_{B^p}(\overline{m}E)^{\frac{1}{q}} \leqq C(\overline{m}E)^{\frac{1}{q}}$$

and this tends to 0 for $\overline{m}E \to 0$. Hence the condition from Theorem B is fulfilled so that our series is the Fourier series of a B^1-a. p. function h. Thus, by **1**, 8. we have $\|h - \sigma_m\|_{B^1} \to 0$ for $m \to \infty$, in particular $\|\sigma_m - \sigma_n\|_{B^1} \to 0$ for $m, n \to \infty$.

We shall now determine a B^1-limit function g from pieces of the σ_m as indicated on the following figure



and show that we can determine $0 = T_0 < T_1 < T_2 < \cdots \to \infty$ so that $\|g\|_{B^p} < \infty$.

From

$$\|\sigma_m\|_{B^p}^p = \overline{\lim_{T\to\infty}}\frac{1}{2T}\int_{-T}^{T}|\sigma_m(x)|^p\,dx < C^p$$

follows

$$\overline{\lim_{|T|\to\infty}}\frac{1}{T}\int_0^T|\sigma_m(x)|^p\,dx < 2C^p.$$

Hence there exists a t_m such that

$$\frac{1}{T}\int_0^T|\sigma_m(x)|^p\,dx < 2C^p \quad \text{for} \quad |T| > t_m,$$

and there exists an $s = s(T_{m-1})$ such that

$$\frac{1}{T-(\pm T_{m-1})}\int_{\pm T_{m-1}}^{T}|\sigma_m(x)|^p\,dx < 2C^p \quad \text{for} \quad \pm T > s(T_{m-1}),$$

respectively, $m = 1, 2, \cdots$. Besides choosing $T_m > t(T_0, \cdots, T_{m-1})$ which by **1**, 1. secures the B^1-convergence of σ_m towards g we choose $T > t_{m+1}$ and $T_m > s(T_{m-1})$ for $m = 1, 2, \cdots$. Then for $T_m \leqq T < T_{m+1}$ we get

$$\frac{1}{T}\int_0^T |g(x)|^p\,dx = \frac{1}{T}\left[\int_0^{T_1}|\sigma_1(x)|^p\,dx + \cdots + \int_{T_{m-1}}^{T_m}|\sigma_m(x)|^p\,dx + \right.$$

$$\left.\int_{T_m}^{T}|\sigma_{m+1}(x)|^p\,dx\right] \leqq \frac{1}{T}\left[2\,C^p(T_1-0) + 2\,C^p(T_2-T_1) + \cdots\right.$$

$$\left.+\,2\,C^p(T_m-T_{m-1})\right] + \frac{1}{T}\int_0^T|\sigma_{m+1}(x)|^p\,dx < 2\,C^p + 2\,C^p = 4\,C^p.$$

For $-T_{m+1} < T \leqq -T_m$ we get analogously that

$$\frac{1}{T}\int_0^T |g(x)|^p\,dx < 4\,C^p.$$

Hence $\| g \|_{B^p} \leqq 4^{\frac{1}{p}}\,C < \infty$, as desired.

We have seen that our series is the Fourier series of a B^1-a. p. point which contains the B^p-bounded function g. It follows by the theorem in **1**, 4. that the B^1-a. p. point contains a B^p-a. p. function f. Thus our series is the Fourier series of a B^p-a. p. function f. This completes the proof of Theorem 1.

Corollary. *Let* $f \sim \sum a(\lambda)\,e^{i\lambda x}$ *be a* B^p-a. p. *function for a fixed* p, $1 < p < \infty$, *and let* $\mathbf{M}$ *be an arbitrary module of real numbers. Then the subseries*

$$\sum_{\lambda \in \mathbf{M}} a(\lambda)\,e^{i\lambda x}$$

is the Fourier series of a B^p-a. p. *function* $f^{\mathbf{M}}$.

Proof. Without loss of generality we may assume $\mathbf{M}$ to be denumerable. A Bochner-Fejér sequence σ_m belonging to $\mathbf{M}$ of the original series $\sum a(\lambda)\,e^{i\lambda x}$ is plainly a Bochner-Fejér sequence belonging to $\mathbf{M}$ of the subseries $\sum_{\lambda\in\mathbf{M}} a(\lambda)\,e^{i\lambda x}$, and for this latter series it is a full Bochner-Fejér sequence.

From **1**, 8. it follows that $\| \sigma_m \|_{B^p} \leqq \| f \|_{B^p}$. This implies, by Theorem 1, that $\sum_{\lambda\in\mathbf{M}} a(\lambda)\,e^{i\lambda x}$ is the Fourier series of a B^p- a. p. function, q. e. d.

Remark. The Corollary is also true when $p = 1$. (Cf. Doss [8], p. 91). The Corollary in the case $p = \infty$ is an immediate consequence of Theorem A.

It is natural to mention, in connection with the above theorems, the following theorem of Pitt, [10], pp. 144—147, which generalizes the Hausdorff-Young Theorem for ordinary Fourier series.

Theorem C. *Let p and q be fixed numbers with $1/p + 1/q = 1$ and $1 < q \leqq 2 \leqq p < \infty$.*

(a) *If f is a B^q-a. p. function with the Fourier series $\sum a_n e^{i\lambda_n x}$ we have*

$$\left(\sum |a_n|^p\right)^{1/p} \leqq \|f\|_{B^q}.$$

(b) *Every trigonometric series $\sum a_n e^{i\lambda_n x}$ with $\sum |a_n|^q < +\infty$ is the Fourier series of a B^p-a. p. function f, and*

$$\|f\|_{B^p} \leqq \left(\sum |a_n|^q\right)^{1/q}.$$

Part (b) of this theorem will be applied in the following.

3. The dual space of the B^p-a. p.-M space, $1 \leqq p < \infty$.

Theorem 2. *Let **M** be a module of real numbers and p, q fixed numbers, $1 < p < \infty$, $1 < q < \infty$, satisfying $1/p + 1/q = 1$. Let further g be a B^q-a. p.—**M** function. Then there exists a B^p-a. p.—**M** function f such that*

$$|M\{f\bar{g}\}| = \|f\|_{B^p} \|g\|_{B^q}. \tag{2}$$

*Thus, when $A_g f = M\{f\bar{g}\}$ is considered as a bounded linear functional on the B^p-a. p.—**M** space, the norm of A_g is equal to $\|g\|_{B^q}$.*

Proof. As shown in 1. 5. the function $f = |g|^{\frac{q}{p}} \operatorname{sign} g$ is a B^p-a. p.—**M** function. An immediate calculation shows that it satisfies (2).

Theorem 3. *Let **M** be a module of real numbers. Let further g be a B^∞-a. p.—**M** function. Then, when f runs over all B^1-a. p.—**M** functions and z runs over all B^∞-zero functions we have*

$$\sup_f \frac{|M\{f\bar{g}\}|}{\|f\|_{B^1}} = \lim_{q\to\infty} \|g\|_{B^q} = \min_z \sup_x |g(x) + z(x)|. \tag{3}$$

*In other words: When $A_g f = M\{f\bar{g}\}$ is considered as a bounded linear functional on the B^1-a. p.—**M** space, the norm of A_g is equal to*

$$\lim_{q \to \infty} \| g \|_{B^q} = \min_z \sup_x |g(x) + z(x)|.$$

Proof. We shall begin by showing that the second sign of equality in (3) is valid and do this in two steps, one for $\leqq$ and one for $\geqq$.

In order to show the inequality $\leqq$, we shall prove that for any B^∞-zero function z we have

$$\lim_{q \to \infty} \| g \|_{B^q} \leqq \sup_x |g(x) + z(x)|.$$

Without changing the value of the left-hand side we can replace g by $g + z$ whereafter the inequality is clear.

In order to show the inequality $\geqq$ (and the existence of the minimum) we have to construct a B^∞-zero function z such that

$$\lim_{q \to \infty} \| g \|_{B^q} \geqq \sup_x |g(x) + z(x)|.$$

Let σ_m be a full Bochner-Fejér sequence of g. Then $\|\sigma_m\|_{B^q} \leqq \|g\|_{B^q}$ for $1 \leqq q < \infty$, so that

$$\lim_{q \to \infty} \| g \|_{B^q} \geqq \lim_{q \to \infty} \| \sigma_m \|_{B^q} = \sup_x |\sigma_m(x)| \tag{4}$$

(for the last sign of equality, see [3], pp. 110—111). We construct now by **1**, 1. a B^1-limit function f from pieces of the σ_m. Then it follows from (4) that

$$\lim_{q \to \infty} \| g \|_{B^q} \geqq \sup_x |f(x)|$$

and obviously $f = g + z$ where z is a B^∞-zero function.

Thus the second sign of equality in (3) is established.

We shall now prove that the sign $\leqq$ holds between the first and the third term in (3). When f is a B^1-a. p. function, g a B^∞-a. p. function, and z a B^∞-zero function we have $M\{f\bar{g}\} = M\{f\overline{(g+z)}\}$, which we obtain from $M\{(f)_n \bar{g}\} = M\{(f)_n \overline{(g+z)}\}$ when we let $n \to \infty$ (see end of **1**, 6). It follows that

$$\frac{M\{f\bar{g}\}}{M\{|f|\}} = \frac{M\{f\overline{(g+z)}\}}{M\{|f|\}} \leqq \sup_x |g(x) + z(x)|.$$

Finally we shall show that the sign $\geq$ holds between the first and the second term in (3). On account of Theorem 2 there exists for every p, $1 < p < \infty$, a B^p-a. p.—**M** (and hence B^1-a. p.—**M**) function f_p such that when $1/p + 1/q = 1$

$$\left| M\{ f_p \bar{g} \} \right| = \| f_p \|_{B^p} \| g \|_{B^q} \geq \| f_p \|_{B^1} \| g \|_{B^q}.$$

Hence

$$\sup_p \frac{\left| M\{ f_p \bar{g} \} \right|}{\| f_p \|_{B^1}} \geq \lim_{q \to \infty} \| g \|_{B^q}.$$

This completes the proof of Theorem 3.

It will now be natural to introduce in the set of B^∞-a. p. functions the norm

$$\| f \|_{B^\infty} = \lim_{q \to \infty} \| f \|_{B^q} = \min_z \sup_x \left| f(x) + z(x) \right|$$

where z runs through all B^∞-zero functions. Obviously a B^∞-zero function z may be characterized as a B^∞-a. p. function with $\| z \|_{B^\infty} = 0$. Now in the usual fashion we introduce B^∞-a. p. points and organize them as a linear metric space. That the B^∞-a. p. space is complete, and hence a Banach space, may for instance be deduced from Theorem 4, below. For an arbitrary module **M** of real numbers we define in the usual way B^∞-a. p.—**M** functions and points. The subspace of B^∞-a. p.—**M** points is called the B^∞-a. p.—**M** space. It is linear and closed and hence a Banach space.

In the following theorem the term "isomorphic mapping" designates "linear one-to-one isometric mapping".

Theorem 4. Main Theorem. *Let* **M** *be an arbitrary module of real numbers and p, q two numbers,* $1 \leq p < \infty$, $1 < q \leq \infty$, *satisfying* $1/p + 1/q = 1$. *Then the dual space of the* B^p-a. p.—**M** *space is isomorphic to the* B^q-a. p.—**M** *space. The isomorphic mapping is given by* $A_g \to g$ *where* A_g *is the bounded linear functional on the* B^p-a. p.—**M** *space given by* $A_g f = M\{ f\bar{g} \}$.

Proof. On account of Theorem 2 and Theorem 3 it suffices to show that every bounded linear functional on the B^p-a. p.—**M** space has the form $A_g f = M\{ f\bar{g} \}$ where g is a B^q-a. p.—**M** function.

In the case $p = 1$ this statement was proved by Doss [8] and was one of the main results of his interesting paper. With the previous preparations at our disposal we can easily treat the general case by his method.

We consider first the case where **M** is a denumerable module of real numbers $\lambda_1, \lambda_2, \cdots$. We put

$$A(e^{i\lambda_n x}) = \bar{a}_n$$

and form the trigonometric series

$$\sum a_n e^{i\lambda_n x}. \tag{5}$$

Let

$$k_m(x) = \sum d_n^{(m)} e^{i\lambda_n x}$$

be a sequence of Bochner-Fejér kernels belonging to **M** (see **1**, 7.). Then if

$$f \sim \sum b_n e^{i\lambda_n x}$$

is an arbitrary B^p-a. p.—**M** function the sequence

$$\sigma_m(x) = \sum d_n^{(m)} b_n e^{i\lambda_n x}$$

is a Bochner-Fejér sequence belonging to **M** of f. Further

$$A\sigma_m = \sum d_n^{(m)} b_n \bar{a}_n.$$

We shall show below that (5) is the Fourier series of a B^q-a. p. —**M** function g. Then we get by **1**, 9.

$$A\sigma_m = \sum d_n^{(m)} b_n \bar{a}_n = M\{\sigma_m \bar{g}\}.$$

Further, from $\|f - \sigma_m\|_{B^p} \to 0$ we get $M\{\sigma_m \bar{g}\} \to M\{f\bar{g}\}$ and $A\sigma_m \to Af$. Hence

$$Af = \lim_{m\to\infty} A\sigma_m = \lim_{m\to\infty} M\{\sigma_m \bar{g}\} = M\{f\bar{g}\},$$

as was to be proved.

That (5) really is the Fourier series of a B^q-a. p.—**M** function is seen in the following way. The sequence

$$\tau_m(x) = \sum d_n^{(m)} a_n e^{i\lambda_n x}$$

is a Bochner-Fejér sequence belonging to **M** of (5). We consider the sequence of bounded linear functionals A_m on the B^p-a. p.—**M** space given by

$$A_m f = A\sigma_m = \sum d_n^{(m)} b_n \bar{a}_n = M\{f\bar{\tau}_m\}.$$

Since $A\sigma_m \to Af$, the sequence A_m converges weakly to A in the Banach space in question. This implies, by the theorem in **1**. 10, the existence of a constant C such that

$$\| A_m \| \leq C \text{ for all } m.$$

From Theorem 2 and Theorem 3 it follows that $\| A_m \| = \| \tau_m \|_{B^q}$. In the special case $q = \infty$ we have

$$\| \tau_m \|_{B^\infty} = \lim_{p \to \infty} \| \tau_m \|_{B^p} = \sup_x |\tau_m(x)|.$$

Thus in the case $1 < q < \infty$ it follows from Theorem 1 and in the case $q = \infty$ from Theorem A that (9) is the Fourier series of a B^q-a. p. function. Of course this function is a B^q-a. p.—**M** function. This completes the proof of the Main Theorem in the case of a denumerable module **M**.

We now pass to the case of an arbitrary module **M**. Let

$$A(e^{i\lambda x}) = \bar{a}_\lambda$$

for $\lambda \varepsilon$ **M**. We shall first show that there exists only a finite number of λ's with $|a_\lambda| > a$ when a is a positive constant. We do this indirectly by assuming that there exists an infinite sequence $\lambda_1, \lambda_2, \cdots$ with $|a_{\lambda_n}| > a$. Let k_n be a sequence of positive numbers with $\sum k_n = \infty$ and if $p \leq 2$ such that $\sum k_n^2 < \infty$ and if $p \geq 2$ such that $\sum k_n^{\frac{p}{p-1}} < \infty$. By Theorem C this implies, since $|a_{\lambda_n}| \leq \| A \|$, that

$$\sum k_n a_{\lambda_n} e^{i\lambda_n x}$$

is the Fourier series of a B^2-a. p.—**M** function for $p \leq 2$ and of a B^p-a. p.—**M** function for $p \geq 2$ — and thus in any case of a B^p-a. p.—**M** function.

Let σ_m be a Bochner-Fejér sequence of f corresponding to the module generated by $\lambda_1, \lambda_2, \cdots$. Then

$$\sigma_m(x) = \sum c_n^{(m)} k_n a_{\lambda_n} e^{i\lambda_n x}$$

where $0 \leq c_n^{(m)} \leq 1$ and $c_n^{(m)} \to 1$ for fixed n and $m \to \infty$. Thus

$$A\sigma_m = \sum c_n^{(m)} k_n |a_{\lambda_n}|^2 \geq a^2 \sum c_n^{(m)} k_n \to \infty$$

for $m \to \infty$. Since $\sigma_m \overset{B^p}{\to} f$ implies $A\sigma_m \to Af$, we have obtained a contradiction.

In particular we have shown that there exists only a denumerable number of λ's with $a_\lambda \neq 0$. The denumerable module generated by these λ's is denoted by $\mathbf{M}_1$ and the elements of this module by $\lambda_1, \lambda_2, \cdots$.

When we consider the contraction of A to the B^p-a. p.—$\mathbf{M}_1$ space, we conclude from the case of a denumerable module treated above that

$$\sum a_{\lambda_n} e^{i\lambda_n x}$$

is the Fourier series of a B^q-a. p.—$\mathbf{M}_1$ function g and that for any B^p-a. p.—$\mathbf{M}_1$ function we have $Af = M\{f\bar{g}\}$.

Now let f be a B^p-a. p.—$\mathbf{M}$ function whose Fourier exponents do not belong to $\mathbf{M}_1$. Then $Af = 0$ since f can be B^p-approximated by trigonometric polynomials without exponents in $\mathbf{M}_1$. From 1, 9. we see that $M\{f\bar{g}\} = 0$. Hence also in this case we get $Af = M\{f\bar{g}\}$.

Finally, let f be an arbitrary B^p-a. p.—$\mathbf{M}$ function. Then by the Corollary and the Remark, p. 11, we can write f in the form

$$f = f^{\mathbf{M}_1} + (f - f^{\mathbf{M}_1})$$

where $f^{\mathbf{M}_1}$ is a B^p-a. p.—$\mathbf{M}_1$ function and $f - f^{\mathbf{M}_1}$ is a B^p-a. p.—$\mathbf{M}$ function whose Fourier exponents do not belong to $\mathbf{M}_1$. From the two special cases just treated we get

$$Af = A(f^{\mathbf{M}_1}) + A(f - f^{\mathbf{M}_1}) =$$

$$M\{f^{\mathbf{M}_1}\bar{g}\} + M\{(f - f^{\mathbf{M}_1})\bar{g}\} = M\{f\bar{g}\}.$$

This completes the proof of the Main Theorem.

Part II.

1. Bohr compactification of the real axis.

Let $\mathbf{M}$ be an arbitrary module of real numbers. We consider the Bohr compactification $R'_{\mathbf{M}}$ of the group R of real numbers with usual topology by all (ordinary) a. p.—$\mathbf{M}$ functions. See e. g. [1], pp. 477—478.

We denote by H the subgroup of R which consists of the x for which $e^{i\lambda x} = 1$ for all $\lambda \varepsilon \mathbf{M}$. In the uninteresting case when $\mathbf{M} = \{0\}$ we have $H = R$. If $\mathbf{M}$ has the form $\{n\xi \mid n = 0, \pm 1, \cdots\}$, we have $H = \left\{n \frac{2\pi}{\xi} \,\middle|\, n = 0, \pm 1, \cdots\right\}$. Otherwise $H = \{0\}$.

We shall make use of the following facts concerning $R'_{\mathbf{M}}$ (see the above quotation).

1) $R'_{\mathbf{M}}$ is a compact abelian group.

2) When the groups $R'_{\mathbf{M}}$ and R/H are considered without their topologies, the group R/H is a subgroup of $R'_{\mathbf{M}}$. The set R/H lies everywhere dense in $R'_{\mathbf{M}}$. Incidentally, in the case $H = \left\{n \frac{2\pi}{\xi} \,\middle|\, n = 0, \pm 1, \cdots\right\}$, the group $R'_{\mathbf{M}}$ is identical with the topological group R/H.

3) When a continuous function on $R'_{\mathbf{M}}$ is contracted to R/H and the contracted function is extended by periodicity with H as periodicity module to R, the resulting function is an a. p.—$\mathbf{M}$ function on R. Conversely, an a. p.—$\mathbf{M}$ function φ on R has H as a periodicity module and may therefore be considered as a function on R/H, and this function extends itself in unique fashion by continuity in $R'_{\mathbf{M}}$ to a continuous function φ' on $R'_{\mathbf{M}}$. This correspondence $\varphi \leftrightarrow \varphi'$ between the a. p.—$\mathbf{M}$ functions on R and the continuous functions on $R'_{\mathbf{M}}$ is of main importance in the following.

4) When $\varphi = e^{i\lambda x}$, $\lambda \varepsilon \mathbf{M}$, the function φ' is a continuous character on $R'_{\mathbf{M}}$. All continuous characters on $R'_{\mathbf{M}}$ can be obtained in this way. [Thus the module $\mathbf{M}$ with discrete topology is the character group of $R'_{\mathbf{M}}$.]

5) Let M denote the Bohr mean and M_N the von Neumann mean, both in R, and let $\int$ denote the Haar integral in $R'_{\mathbf{M}}$ with $\int 1 = 1$. Then for any a. p.—**M** function φ on R we have

$$M\varphi = M_N\varphi = \int \varphi'.$$

2. Extension of the correspondence $\varphi \leftrightarrow \varphi'$ between the a. p. —M functions on R and the continuous functions on $R'_{\mathbf{M}}$ to a correspondence between the B^p-a. p.—M space over R and the space L^p over $R'_{\mathbf{M}}$, $1 \leq p \leq \infty$.

As well-known for any fixed p, $1 \leq p < \infty$, the set of measurable p-integrable functions $g(x')$ on $R'_{\mathbf{M}}$ is organized as a Banach space L^p by the norm

$$\|g\|_p = \left(\int |g(x')|^p\right)^{\frac{1}{p}}$$

while for $p = \infty$ the set of essentially bounded measurable functions $g(x')$ on $R'_{\mathbf{M}}$ is organized as a Banach space L^∞ by the norm

$$\|g\|_\infty = \lim_{p \to \infty} \|g\|_p = \operatorname{vrai}\max_{x'} |g(x')|.$$

Functions which are equal almost everywhere (a. e.) are considered to be the same function.

For this, and also for results used in the following, we refer the reader to Loomis' book [9], Chapter III, pp. 29—47.

We shall now prove the following

Correspondence Theorem. *Let* **M** *be a module of real numbers. Then there exists a mapping $f \to f'$ of the set of B^1-*a. p.—**M** *functions on the set of integrable functions on $R'_{\mathbf{M}}$ which is an extension of the previous mapping $\varphi \to \varphi'$ of the set of* a. p.—**M** *functions on the set of continuous functions on $R'_{\mathbf{M}}$. This mapping has the following properties.*

1. *For any fixed p, $1 \leq p \leq \infty$, the contraction of the mapping to the set of B^p-*a. p.—**M** *functions is a linear isometric mapping of the set of B^p-*a. p.—**M** *functions on the space L^p over $R'_{\mathbf{M}}$. It may be considered as a one-to-one linear isometric mapping of the B^p-*a. p.—**M** *space on the space L^p over $R'_{\mathbf{M}}$.*

2. *For any* B^1-a. p.—**M** *function* f *we have*

$$|f|' = |f'|, \quad (\Re f)' = \Re(f'), \quad (\Im f)' = \Im(f'),$$
$$(\bar{f})' = \overline{(f')}, \quad ((f)_n)' = (f')_n$$

and when f *is real*

$$(f^+)' = (f')^+ \quad \text{and} \quad (f^-)' = (f')^-.$$

3. *If* f *is a* B^p-a. p.—**M** *function and* g *is a* B^q-a. p.—**M** *function,* $1/p + 1/q = 1$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, *then*

$$(fg)' = f'g'.$$

4. *If the* B^1-a. p.—**M** *function* f *has the Fourier series* $\sum a_n e^{i\lambda_n x}$, *then* f' *has the Fourier series*

$$\sum a_n (e^{i\lambda_n x})',$$

in particular $Mf = \int f'$.

5. *If* f *is a* B^p-a. p.—**M** *function for a fixed* p, $1 \leq p < \infty$, *and* $1 \leq q < \infty$, *then*

$$(|f|^{\frac{p}{q}} \operatorname{sign} f)' = |f'|^{\frac{p}{q}} \operatorname{sign} f'.$$

6. *The asymptotic distribution function of a real* B^1-a. p.—**M** *function* f *is identical with the distribution function of* f'.

Proof. Let f be a B^p-a. p.—**M** function for a fixed p, $1 \leq p < \infty$. Then there exists a sequence of a. p.—**M** functions φ_n such that $\varphi_n \overset{B^p}{\rightarrow} f$. In particular $\| \varphi_m - \varphi_n \|_{B^p} \rightarrow 0$ for $m, n \rightarrow \infty$. Hence by 3) and 5) we get

$$\| \varphi_m' - \varphi_n' \|_p^p = \int | \varphi_m' - \varphi_n' |^p = \int (| \varphi_m - \varphi_n |^p)' = \| \varphi_m - \varphi_n \|_{B^p}^p \rightarrow 0$$

for $m, n \rightarrow \infty$. It follows that φ_n' will p-converge to a function g_p from L^p which is determined a. e. This function g_p depends only on f and not on the sequence φ_n. To see this, let ψ_n be another sequence of a. p.—**M** functions which B^p-converges to f, and suppose that $\psi_n' \overset{p}{\rightarrow} h_p$. Then the combined sequence $\varphi_1, \psi_1, \varphi_2, \psi_2, \cdots$ will B^p-converge to f, and it follows that $\varphi_1', \psi_1', \varphi_2', \psi_2', \cdots$ will p-converge, in particular that $\varphi_n' - \psi_n' \overset{p}{\rightarrow} 0$. Hence $\| h_p - g_p \|_p = 0$.

We see also that if p_1 and p_2 are two values of p for which f is B^p-a. p.—**M**, then $g_{p_1} = g_{p_2}$ a. e., for if $1 \leq p_1 < p_2$ and $\varphi_n \xrightarrow{B^{p_2}} f$, then $\varphi_n \xrightarrow{B^{p_1}} f$, and we get $\varphi_n' \xrightarrow{p_2} g_{p_2}$, $\varphi_n' \xrightarrow{p_1} g_{p_1}$ the first of which implies that $\varphi_n' \xrightarrow{p_1} g_{p_2}$ so that $\| g_{p_2} - g_{p_1} \|_{p_1} = 0$.

If in particular f is an a. p.—**M** function, then $f, f, \cdots \xrightarrow{B^p} f$ and since $f', f', \cdots \xrightarrow{p} f'$ we see that the function g corresponding to f is f'. Also in the general case when f is a B^1-a. p.—**M** function the corresponding function g, defined by the above procedure, will be denoted by f'.

Now let f be a B^∞-a. p.—**M** function. We shall show that f' belongs to L^∞. Let $\sup\limits_x |f(x)| = C$. There exists (I, **1**, 8.) a sequence of a. p.—**M** functions φ_n which B^1-converges to f and has $|\varphi_n(x)| \leq C$. Then $|\varphi_n'(x')| \leq C$ and $\varphi_n' \xrightarrow{1} f'$ which implies $|f'(x')| \leq C$, a. e. Thus f' belongs to L^∞, and furthermore $\operatorname{vrai\,max}\limits_{x'} |f'(x')| \leq \sup\limits_x |f(x)|$.

For a fixed p, $1 \leq p < \infty$, let f be a B^p-a. p.—**M** function. We choose a sequence of a. p.—**M** functions φ_n which B^p-converges to f. Then $\| \varphi_n \|_{B^p} \to \| f \|_{B^p}$, $\| \varphi_n' \|_p \to \| f' \|_p$, and $\| \varphi_n \|_{B^p} = \| \varphi_n' \|_p$ so that

$$\| f \|_{B^p} = \| f' \|_p .$$

If f is a B^1-a. p.—**M** function and φ_n is chosen as usual we get $M\varphi_n \to Mf$, $\int \varphi_n' \to \int f'$, and $M\varphi_n = \int \varphi_n'$ so that

$$Mf = \int f'. \tag{1}$$

Let f and g be two B^1-a. p.—**M** functions. Let φ_n and ψ_n be chosen correspondingly. Then we get successively, when a and b denote complex numbers, $a\varphi_n + b\psi_n \xrightarrow{B^1} af + bg$, $\varphi_n' \xrightarrow{1} f'$, $\psi_n' \xrightarrow{1} g'$, $(a\varphi_n + b\psi_n)' \xrightarrow{1} (af + bg)'$, $a\varphi_n' + b\psi_n' \xrightarrow{1} af' + bg'$. Since $(a\varphi_n + b\psi_n)' = a\varphi_n' + b\psi_n'$ we get

$$(af + bg)' = af' + bg'.$$

In an analogous way be obtain the relations in 2.

In order to prove 3. we consider first the case $1 < p < \infty$, and hence $1 < q < \infty$. That fg is B^1-a. p.—**M** follows from I, **1**, 6. Let φ_n and ψ_n be chosen in the usual way for f and g respectively. Then by I, **1**, 6. we get $\varphi_n \psi_n \xrightarrow{B^1} fg$. Hence we get successively

$\varphi_n' \xrightarrow{p} f'$, $\psi_n' \xrightarrow{q} g'$, $(\varphi_n \psi_n)' \xrightarrow{1} (fg)'$, and $\varphi_n' \psi_n' \xrightarrow{1} f'g'$ (by the result corresponding to I, **1**, 6. for the space L^p). Since $(\varphi_n \psi_n)' = \varphi_n' \psi_n'$ we get

$$(fg)' = f'g'.$$

Next, we consider the case $p = 1$, $q = \infty$. That fg is B^1-a. p.—**M** follows from I, **1**, 6. We showed there that $(f)_n g \xrightarrow{B^1} fg$. Hence $((f)_n g)' \xrightarrow{1} (fg)'$ since our mapping is linear and isometric. However, $(f)_n$ and g are B^∞-a. p.—**M**, in particular B^2-a. p.—**M**. Hence from the case just treated we get $((f)_n g)' = ((f)_n)'g' = (f')_n g'$ and this $\xrightarrow{1} f'g'$ since f' is in L^1 and g' in L^∞. Thus also in this case $(fg)' = f'g'$. This completes the proof of 3.

From the special result (1) the general statement 4. is now an easy consequence.

The proof of 5. is analogous to the proof of 3. It uses I, **1**, 5. instead of I, **1**, 6. However, we shall not use 5. in the following.

Next, for any fixed p, $1 \leq p < \infty$, we consider an arbitrary function $g(x')$ from L^p. There exists a sequence of continuous functions φ_n' on $R_{\mathbf{M}}'$ which $\xrightarrow{p} g$. [As stated in 3) every continuous function on $R_{\mathbf{M}}'$ is of the form φ' where φ is an a. p.—**M** function.] Then $\| \varphi_m - \varphi_n \|_{B^p} = \| \varphi_m' - \varphi_n' \|_p \to 0$ for $m, n \to \infty$ and hence φ_n will B^p-converge to a B^p-a. p.—**M** function f for which $f' = g$. Since the mapping is linear and isometric, the B^p-a. p.—**M** functions f for which $f' = g$ are exactly the functions in a B^p-a. p.—**M** point. Thus the contraction of our mapping $f \to f'$ to the set of B^p-a. p.—**M** functions may be considered as a one-to-one mapping of the B^p-a. p.—**M** space *on* the space L^p. Obviously this mapping is linear and isometric. This proves 1. for $1 \leq p < \infty$.

Next we consider the case $p = \infty$. We have already seen that the set of B^∞-a. p.—**M** functions is mapped into the space L^∞ and that $\| f' \|_\infty \leq \sup_x | f(x) |$.

Now let $g(x')$ be an arbitrary function from L^∞ and let $\| g \|_\infty = C$. Let φ_n be a sequence of a. p.—**M** functions such that $| \varphi_n'(x') | \leq C$ and $\varphi_n' \xrightarrow{1} g$. Then $\| \varphi_m - \varphi_n \|_{B^1} \to 0$ for $m, n \to \infty$ and hence by I, **1**, 1. we can construct a B^1-limit function f_0 of the sequence φ_n from pieces of the φ_n. It follows that $| f_0(x) | \leq C$ and that $f_0' = g$. Since $\sup_x | f_0(x) | \leq \| g \|_\infty$ and since, as we have seen previously, all B^∞-a. p.—**M** functions f which are mapped in g have

$$(2) \qquad \sup_x |f(x)| \geqq \|g\|_\infty ,$$

we see that

$$(3) \qquad \sup_x |f_0(x)| = \|g\|_\infty .$$

The B^∞-a. p.—**M** functions f which are mapped in the same g from L^∞ belong in particular to the same B^1-a. p.—**M** point, and hence they differ from each other by a bounded B^1-zero function, i. e., a B^∞-zero function. Thus they belong to the same B^∞-a. p.—**M** point. All functions in this point are mapped in the function g. Thus the contraction of the mapping $f \to f'$ to the set of B^∞-a. p.—**M** functions can be considered as a one-to-one linear mapping of the B^∞-a. p.—**M** space on the space L^∞. From (2) and (3) we can now conclude that for any B^∞-a. p.—**M** function f we have

$$\|f'\|_\infty = \inf_z \sup_x |f(x) + z(x)| = \|f\|_{B^\infty}$$

where z runs through all B^∞-zero functions. Thus the contracted mapping is isometric. Using that

$$\|f'\|_\infty = \lim_{p \to \infty} \|f'\|_p = \lim_{p \to \infty} \|f\|_{B^p}$$

we get the other expression

$$\|f\|_{B^\infty} = \lim_{p \to \infty} \|f\|_{B^p}$$

for the B^∞-norm. This completes the proof of 1.

Finally we shall prove 6. Since this part of the Correspondence Theorem will not be used in the following we shall treat it shortly. It is known that every real B^1-a. p. function possesses an asymptotic distribution function. We shall use a proof of this theorem which was communicated to the authors of [5] by Jessen and published in [5], pp. 101—103; in order to save space we shall assume that the reader knows the proof and the notations used therein. It is easily seen that the function $\Phi(f(x))$ occurring l. c., p. 102 can be written

$$\Phi(f(x)) = 1 + \frac{(f(x)-\beta)^+ - (f(x)-\alpha)^+}{\beta-\alpha}.$$

Hence

$$(\Phi(f))' = \Phi(f')$$

so that

$$M_B\{\Phi(f)\} = \int \Phi(f').$$

Then 6. is a simple consequence of the inequalities

$$\psi_1(\beta) \geqq M_B\{\Phi(f)\} \geqq \psi(\alpha)$$

which were proved l. c., p. 102 and the corresponding inequalities for the function f'.

This completes the proof of the Correspondence Theorem.

3. Application of the Correspondence Theorem to a proof of the Main Theorem.

If in the Main Theorem we replace the B^p-a. p.—**M** space by the space L^p and the B^q-a. p.—**M** space by the space L^q and the mean value M by the Lebesgue integral $\int$, we obtain a classical result of F. Riesz which is valid even for spaces L^p in the abstract case. A proof of this theorem is given in [9], Chapter III, p. 42 and p. 47.

By use of the Correspondence Theorem we shall now deduce our Main Theorem from F. Riesz's result. Let p be a fixed number, $1 \leqq p < \infty$. It follows easily from the Correspondence Theorem that the mapping

(4) $$A \rightarrow A'$$

where A is a bounded linear functional on the B^p-a. p.—**M** space and A' is the functional on the space L^p over $R'_{\mathbf{M}}$ defined by

$$A'f' = Af$$

is a one-to-one linear isometric mapping of the dual space of the B^p-a. p.—**M** space on the dual space of the space L^p. However, on account of Riesz's theorem the mapping

(5) $$A'_{g'} \rightarrow g'$$

where

$$A'_{g'}(f') = \int f'\overline{(g')}$$

is a one-to-one linear isometric mapping of the dual space of L^p over $R'_{\mathbf{M}}$ on the space L^q over $R'_{\mathbf{M}}$. Finally, by the Correspondence Theorem the mapping

$$g' \to g \tag{6}$$

is a one-to-one linear isometric mapping of the space L^q over $R'_{\mathbf{M}}$ on the B^q-a. p.—$\mathbf{M}$ space and

$$\int f'\overline{(g')} = M\{f\bar{g}\}$$

for every B^p-a. p.—$\mathbf{M}$ function f and every B^q-a. p.—$\mathbf{M}$ function g. The mapping

$$A_g \to g$$

which results from the mappings (4), (5), (6) is then a one-to-one linear isometric mapping of the dual space of the B^p-a. p.—$\mathbf{M}$ space on the B^q-a. p.—$\mathbf{M}$ space, and

$$A_g f = A'_{g'}(f') = \int f'\overline{(g')} = M\{f\bar{g}\}.$$

This completes the proof of the Main Theorem.

Appendix.

The following theorem shows that for a given module $\mathbf{M}$ of real numbers and a fixed p, $1 \leq p < \infty$, the B^p-a. p.—$\mathbf{M}$ space and the W^p-a. p.—$\mathbf{M}$ space have the same dual space.

Theorem. *Let* $\mathbf{M}$ *be a module of real numbers and* p, q *two numbers,* $1 \leq p < \infty$, $1 < q \leq \infty$, *satisfying* $1/p + 1/q = 1$. *Then the dual space of the* W^p-a. p.—$\mathbf{M}$ *space is isomorphic to the* B^q-a. p.—$\mathbf{M}$ *space. The isomorphic mapping is given by* $A_g \to g$ *where* A_g *is the bounded linear functional on the* W^p-a. p.—$\mathbf{M}$ *space given by* $A_g f = M\{f\bar{g}\}$.

Proof. For any W^p-a. p. function f we have $\|f\|_{B^p} = \|f\|_{W^p}$, for by I, **1**, 5. the function $|f(x)|^p$ is W^1-a. p. so that

$$\lim_{T \to \infty} \frac{1}{T} \int_a^{a+T} |f(x)|^p \, dx$$

exists uniformly in a. Further every B^p-a. p.—**M** function may be B^p-approximated by W^p-a. p.—**M** functions, for it may even be B^p-approximated by a. p.—**M** functions. It follows from this that every bounded linear functional on the W^p-a. p.—**M** space extends itself in unique fashion by B^p-continuity to a bounded linear functional on the B^p-a. p.—**M** space with the same norm as the original functional, and that conversely every bounded linear functional on the B^p-a. p.—**M** space induces a bounded linear functional on the W^p-a. p.—**M** space. Our theorem is then a consequence of the Main Theorem.

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